

# Bounds on Separating Redundancy

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**Abstract**—It is known that any linear code allows for a decoding strategy that treats errors and erasures separately over an error-erasure channel. In particular, a carefully designed parity-check matrix makes efficient erasure correction possible through iterative decoding while instantly providing the necessary parity-check equations for independent error correction on demand. The separating redundancy of a given linear code is the number of parity-check equations in a smallest parity-check matrix that has the required property for this error-erasure separation. In a sense, it is a parameter of a linear code that represents the minimum overhead for efficiently separating erasures from errors. While several bounds on separating redundancy are known, there still remains a wide gap between upper and lower bounds except for a few limited cases. In this paper, using probabilistic combinatorics and design theory, we improve both upper and lower bounds on separating redundancy.

**Index Terms**—Linear code, error-erasure channel, separating redundancy, probabilistic method, covering, combinatorial design.

## I. INTRODUCTION

DISCRETE error-erasure channels with input alphabet  $\Gamma$  and output alphabet  $\Gamma \cup \{e\}$ , where  $e \notin \Gamma$ , are the simplest abstract models that combine two different types of fundamental channels. For instance, the most elementary error-erasure channel is the natural combination of a binary symmetric channel and a binary erasure channel in which each bit is independently either flipped to the other symbol with probability  $p_{\text{error}}$ , altered to  $e$  with probability  $p_{\text{erasure}}$ , or kept intact with probability  $1 - p_{\text{error}} - p_{\text{erasure}}$ . Such combined channels have been studied not only because they are natural from the purely theoretical viewpoint, but also because they are reasonable models of noise in various scenarios. Situations in which a combination of additive noise and loss of data occurs include wireless communications with cross-layer protocols [1]–[5], delay-sensitive optical communications [6], [7] including some proposed deep-space communications systems [8], [9], and magnetic and optical recoding [10], [11] among others.

There are various possible decoding algorithms that can handle simultaneous occurrences of errors and erasures. Since the location of each erasure is known to the decoder, one straightforward approach is to assign a random symbol to each erased digit and perform standard error correction. With this

method, when exactly  $l$  digits are erased, a naive decoder for a  $|\Gamma|$ -ary error-correcting code would try  $|\Gamma|^l$  possible patterns to infer the original codeword. Message-passing decoding methods such as belief propagation may also be used by assigning an appropriate likelihood to each erased digit [12].

Recently, an interesting alternative approach was proposed in [13], where errors and erasures are separately corrected by a linear code. They revisited a classical strategy in which the decoder first corrects errors using the punctured code obtained by discarding the erased digits and then handles the erasures with the original linear code. The innovative part of their idea is use of a carefully chosen parity-check matrix that allows for instantly providing parity-check matrices for appropriate punctured codes on demand while making efficient erasure correction possible by iterative decoding.

As we will formally define later, a parity-check matrix is called  $l$ -separating if it admits this on-demand separation for all patterns of  $l$  or fewer erasures. It is not difficult to prove that any linear code of minimum distance  $d$  has an  $l$ -separating parity-check matrix for any  $l \leq d - 1$  if there is no restriction on the number of redundant parity-check equations. However, to minimize overhead for implementation and reduce decoding complexity, we would like our parity-check matrix to be as small as possible.

The  $l$ -separating redundancy of a linear code  $\mathcal{C}$  is the number of parity-check equations in a smallest  $l$ -separating parity-check matrix for  $\mathcal{C}$ . While the  $l$ -separating redundancy of a linear code is important in the study of error-erasure separation, it appears to be quite difficult to give a precise estimate, let alone determine the exact value. Indeed, the precise values are known only for a few limited cases such as for the binary extended Hamming codes with  $l = 1$  [14] and for maximum distance separable (MDS) codes with a few specific  $l$  [15]. For linear codes in general, the current best upper and lower bounds are still far apart [13], [16].

The purpose of this paper is to improve the estimate of the separating redundancy of a linear code through probabilistic combinatorics and design theory. We refine both upper and lower bounds on  $l$ -separating redundancy that work for any linear code. To more sharply bound the parameter from below, we improve the simple volume bound given in [13] to one of Schönheim type [17]. We also give new strong upper bounds using probabilistic combinatorics. In addition to these, a known upper bound based on design theory is refined. While the design theoretic approach is not as universally strong as the probabilistic one, it is shown to give a sharper estimate than any other known bound in some cases. As far as the authors are aware, these are the first substantial general progress towards closing the gap between upper and lower bounds since the introduction of separating redundancy.

In the next section, we briefly review the concept of

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separating redundancy and known bounds. Section III gives our improved lower and upper bounds and explains how our approaches mathematically refine the previously known techniques from a general viewpoint. Numerical examples in Section IV illustrate how our bounds compare against the known general bounds in specific cases. Section V concludes this paper with some remarks.

## II. PRELIMINARIES

In this section, we mathematically define separating redundancy and review known results. While we also define most of the basic notions in coding theory we use, for more comprehensive treatments, we refer the reader to standard textbooks such as [18], [19].

As usual, an  $[n, k, d]_q$  linear code  $\mathcal{C}$  of length  $n$ , dimension  $k$ , and minimum distance  $d$  over the finite field  $\mathbb{F}_q$  of order  $q$  is a  $k$ -dimensional subspace of the  $n$ -dimensional vector space  $\mathbb{F}_q^n$  over  $\mathbb{F}_q$  such that  $\min\{\text{wt}(\mathbf{c}) \mid \mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}\} = d$ , where  $\text{wt}(\mathbf{a})$  for  $\mathbf{a} \in \mathbb{F}_q^n$  is the Hamming weight of  $\mathbf{a}$ . Each vector in  $\mathcal{C}$  is a *codeword* of the linear code.

The *dual code*  $\mathcal{C}^\perp$  of the linear code  $\mathcal{C}$  is the Euclidean dual space of  $\mathcal{C}$ , that is,  $\mathcal{C}^\perp = \{\mathbf{c} \in \mathbb{F}_q^n \mid \mathbf{c} \cdot \mathbf{c}' = \mathbf{0} \text{ for any } \mathbf{c}' \in \mathcal{C}\}$ . The *dual distance*  $d^\perp$  of  $\mathcal{C}$  is the minimum distance of its dual code  $\mathcal{C}^\perp$ . Regarding each element of  $\mathbb{F}_q^n$  as an  $n$ -dimensional row vector over  $\mathbb{F}_q$ , a *parity-check matrix*  $H$  for  $\mathcal{C}$  is an  $m \times n$  matrix over  $\mathbb{F}_q$  whose rows span  $\mathcal{C}^\perp$ . A *supercode*  $\mathcal{D}$  of  $\mathcal{C}$  is a set  $\mathcal{D} \subseteq \mathbb{F}_q^n$  such that  $\mathcal{C} \subseteq \mathcal{D}$ . If  $\mathcal{D}$  is also a linear code, its parity-check matrix consists of some, but not necessarily all, codewords of  $\mathcal{C}^\perp$  and is of rank at most  $n - k$  over  $\mathbb{F}_q$ .

We use the nonnegative integers less than  $n$  to specify the coordinates of the vector space  $\mathbb{F}_q^n$ . For a set  $S \subset \{0, 1, \dots, n-1\}$  of coordinates, the *punctured code*  $\mathcal{C}_{\overline{S}}$  of the  $[n, k, d]_q$  linear code  $\mathcal{C}$  by  $S$  is the  $[n - |S|, k', d']_q$  linear code for some  $k' \leq k$  and some  $d' \leq d$  obtained by deleting the coordinates in  $S$  from the codewords of  $\mathcal{C}$ . In other words,  $\mathcal{C}_{\overline{S}}$  is the linear code obtained by *puncturing*  $\mathcal{C}$  on  $S$ .

Assume that the sender transmitted a codeword of an  $[n, k, d]_q$  linear code  $\mathcal{C}$  and that the channel introduced to the message exactly  $|S|$  erasures on the coordinates  $S \subset \{0, 1, \dots, n-1\}$  along with some errors elsewhere in the same  $n$ -digit message block. While there can be multiple ways to overcome a combination of errors and erasures, one of the simplest methods is to correct the errors by the punctured code  $\mathcal{C}_{\overline{S}}$  and then the erasures by  $\mathcal{C}$ . It is well-known that for any fixed integer  $z \geq 0$ , this error-erasure separation strategy provides a decoding algorithm that corrects up to  $x$  errors and up to  $y$  erasures if  $2x + y \leq d - 1 - z$  and declares detection of an anomaly if  $z \geq 1$  and  $d - z \leq 2x + y \leq d - 1$ , achieving the limit of error-erasure correction capability dictated by the principle of minimum distance decoding (see, for example, [19]).

For the above decoding method to be effective, however, we need to be able to quickly provide parity-check matrices for appropriate punctured codes on the fly. It is shown in [13] that this can be done with a special parity-check matrix.

Let  $H$  be a parity-check matrix for a given  $[n, k, d]_q$  linear code  $\mathcal{C}$ . Define  $H(S)$  to be the submatrix of  $H$  obtained by

discarding all rows that contain a nonzero element in at least one coordinate in  $S$  and deleting all columns corresponding to the coordinates in  $S$ . The submatrix  $H(S)$  is a parity-check matrix for a supercode of  $\mathcal{C}_{\overline{S}}$ . The parity-check matrix  $H$  is *S-separating* if  $H(S)$  is a parity-check matrix for  $\mathcal{C}_{\overline{S}}$ . We call  $H$  *l-separating* if it is *S-separating* for any subset  $S \subset \{0, 1, \dots, n-1\}$  of cardinality less than or equal to  $l$ .

To give a concrete example, consider the following parity-check matrix  $H$  for the  $[8, 4, 4]_2$  extended Hamming code.

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

For  $S = \{6, 7\}$ , we have

$$H(S) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

which is a correct parity-check matrix for the  $[6, 4, 2]_2$  linear code obtained by puncturing the extended Hamming code on the last two bits. Hence, this particular  $H$  is  $\{6, 7\}$ -separating. However, it is not 2-separating because, for example, if  $S = \{5, 6\}$ , then we have

$$H(S) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

which does not have enough linearly independent rows to be a valid parity-check matrix for the punctured code obtained by deleting the fifth and sixth coordinates.

Roughly speaking, an *l-separating* parity-check matrix has valid parity-check matrices for all required punctured codes ready for use as conveniently stored submatrices under the assumption that at most  $l$  erasures can happen. Indeed, for any pattern of  $l$  or fewer erasures, a parity-check matrix for the corresponding punctured code can be obtained by taking the rows that do not check any of the erased digits and then throwing away the zeros at the erased positions.

Interestingly, if  $l$  is smaller than the minimum distance  $d$  of a given linear code  $\mathcal{C}$ , it can be shown that for any set  $S$  of coordinates of cardinality  $|S| \leq l$ , an *l-separating* parity-check matrix for  $\mathcal{C}$  contains a row that has exactly one nonzero element in the coordinates in  $S$  [13]. In other words, for any pattern of  $l$  or fewer erasures, it always contains a row that checks exactly one erased coordinate, so that erasures can be corrected quickly one-by-one without solving a system of linear equations.

Trivially, for  $l \geq 1$ , an *l-separating* parity-check matrix is also  $(l-1)$ -separating. It is also straightforward to see that if  $|S| \leq d-1$ ,  $H(S)$  is a parity-check matrix for  $\mathcal{C}_{\overline{S}}$  if and only if  $\text{rank}(H(S)) = n - k - |S|$  over  $\mathbb{F}_q$ . Note that if  $|S| \geq d$ , the dimension of  $\mathcal{C}_{\overline{S}}$  can be less than  $k$ , in which case the erasure pattern that corresponds to  $S$  can not be corrected by  $\mathcal{C}$ . Hence, in what follows, as in the original study of separating redundancy in [13], we consider *l-separation* for  $l \leq d-1$ .

In general, all else being equal, it is more desirable for an *l-separating* parity-check matrix to have fewer rows. However, even for a small parity-check matrix  $H$ , it can be a daunting

task to check whether all possible  $H(S)$  are valid parity-check matrices for  $\mathcal{C}_{\overline{S}}$ . The following proposition makes it easier to verify that a given parity-check matrix is  $l$ -separating.

**Proposition 2.1 ([13]):** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. Then, for any  $l \leq \min\{d, n - k\} - 1$ , a parity-check matrix for  $\mathcal{C}$  is  $l$ -separating if and only if it is  $S$ -separating for any  $S \subset \{0, 1, \dots, n - 1\}$  of cardinality  $l$ .

The above proposition says that for virtually all cases, we only need to check the  $\binom{n}{l}$  patterns of exactly  $l$  erasures rather than all possible patterns of up to  $l$  erasures. The only exceptional case is  $(d - 1)$ -separation of an  $[n, k, d]_q$  linear code with  $d = n - k + 1$ , that is,  $(d - 1)$ -separation of a *maximum distance separable (MDS)* code. In this unique exceptional case, the situation is even simpler.

**Proposition 2.2 ([13]):** Any parity-check matrix for any  $[n, k, n - k + 1]_q$  linear code is  $S$ -separating for any  $S \subset \{0, 1, \dots, n - 1\}$  with  $|S| = n - k$ .

Thus, for MDS codes of minimum distance  $d$ , we only need to consider the case  $l \leq d - 2$ , where Proposition 2.1 is applicable the same way as in any other linear code.

A crucial problem regarding the error-erasure separation method is how small an  $l$ -separating parity-check matrix can be. The  *$l$ -separating redundancy*  $s_l(\mathcal{C})$  of an  $[n, k, d]_q$  linear code  $\mathcal{C}$  is the number of rows in a smallest  $l$ -separating parity-check matrix for  $\mathcal{C}$ . Note that if  $\mathcal{C}$  is an MDS code, by Proposition 2.2, we have  $s_{d-1}(\mathcal{C}) = s_{d-2}(\mathcal{C})$ . For this reason, we focus on the case  $l \leq \min\{d, n - k\} - 1$ , that is,  $l \leq d - 2$  for an MDS code and  $l \leq d - 1$  for any other linear code.

As far as the authors are aware, the following is the only known nontrivial lower bound on separating redundancy for a general linear code.

**Theorem 2.3 ([13]):** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code with dual distance  $d^\perp$ . For any  $l \leq \min\{d, n - k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \geq \frac{\binom{n}{l}(n - k - l)}{\binom{n - d^\perp}{l}}.$$

While the above bound can be proved by a simple volume argument, it is nonetheless best possible in the sense that some linear codes do achieve it for some  $l$  [14], [15].

There are several known general upper bounds on  $s_l(\mathcal{C})$  that are of comparable usefulness. Among them, the one based on the pigeonhole principle is often the sharpest, especially when the parameter  $l$  is not too large and no structural information about  $\mathcal{C}$  except its basic code parameters is available.

To describe the upper bound, define

$$f_q(a, b) = \sum_{i=0}^a (-1)^i \binom{a}{i} \prod_{j=0}^{b-1} (q^{a-i} - q^j),$$

where  $a$  and  $b$  are nonnegative integers and  $q$  is a prime power.

**Theorem 2.4 ([13]):** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n - k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  is less than or equal to the minimum integer  $t$  that satisfies

$$\sum_{i=0}^t \binom{t}{i} f_q(i, l) \frac{\prod_{j=0}^{n-k-l-1} (q^t - q^{i+j})}{\prod_{h=0}^{n-k-1} (q^t - q^h)} > 1 - \frac{1}{\binom{n}{l}}.$$

A weaker version of the above bound in closed form is also available in [13].

The following is another general bound proved by extending the idea of *generic erasure-correcting sets* [20]–[22].

**Theorem 2.5 ([13]):** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n - k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \leq \sum_{i=1}^{l+1} \binom{n-k}{i} (q-1)^{i-1}.$$

While the right-hand side of the inequality in the above theorem is quite large compared to the lower bound in Theorem 2.3, it was proved by a constructive method and gives an upper bound in simple form. In addition, it is typically the sharpest when Theorem 2.4 becomes loose due to large  $l$ .

The above general upper bounds depend on the alphabet size  $q$  of a linear code, while the lower bound given in Theorem 2.3 does not. Interestingly, there is also one known general upper bound that does not depend on  $q$ .

To describe the bound, we need a special combinatorial design. In what follows,  $\mathbb{N}$  represents the set of positive integers. Let  $n, \mu, l, \lambda \in \mathbb{N}$  such that  $n \geq \mu \geq l$ . An  $l$ -( $n, \mu, \lambda$ ) *covering* is a pair  $(V, \mathcal{B})$  of a finite set  $V$  of cardinality  $n$  and a collection  $\mathcal{B}$  of  $\mu$ -subsets of  $V$  such that every  $l$ -subset of  $V$  appears in at least  $\lambda$  elements of  $\mathcal{B}$ . The *covering number*  $C_\lambda(n, \mu, l)$  is the cardinality  $|\mathcal{B}|$  of a smallest  $\mathcal{B}$  such that there exists an  $l$ -( $n, \mu, \lambda$ ) covering  $(V, \mathcal{B})$ .

**Theorem 2.6 ([13]):** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n - k\} - 1$ , define  $L = \{i \in \mathbb{N} \mid l \leq i \leq \min\{d, n - k\} - 1\}$ . The  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \leq \min_{\mu \in \mathbb{N}} \left\{ (n - k - \mu) C_1(n, \mu, l) + \binom{n}{l} (\mu - l) \right\}.$$

The covering number  $C_1(n, \mu, l)$  has extensively been investigated in combinatorics. In particular, using a strong probabilistic method known as the *Rödl nibble* [23], it was shown that  $C_1(n, \mu, l)$  is asymptotically  $\binom{n}{l} / \binom{\mu}{l}$ .

**Theorem 2.7 ([24]):** For fixed integers  $2 \leq l \leq \mu$ ,

$$\frac{\binom{n}{l}}{\binom{\mu}{l}} \leq C_1(n, \mu, l) \leq (1 + o(1)) \frac{\binom{n}{l}}{\binom{\mu}{l}},$$

where the  $o(1)$  term tends to zero as  $n$  tends to infinity.

For a comprehensive list of results on bounds on and known exact values of  $C_1(n, \mu, l)$ , we refer the reader to [25] (see also [26]–[28] for the more recent results not covered in the list).

There are several other bounds on separating redundancy that consider very special cases such as 1-separation for cyclic codes and linear codes with specific codewords. One may also derive an upper bound for a linear code if its weight distribution is partially known. For those specialized bounds, the interested reader is referred to [13], [16].

### III. NEW BOUNDS

This section is divided into three subsections to present our bounds on separating redundancy in an organized manner. Section III-A refines the lower bound in Theorem 2.3. Our probabilistic upper bounds are given in Section III-B. Section III-C proves a design theoretic upper bound that improves Theorem 2.6.



### A. Improved lower bound

While the lower bound in Theorem 2.3 is achievable in some cases, a simple observation shows that using the idea of a covering leads to a tighter bound.

We first generalize the concept of a covering  $(V, \mathcal{B})$  by allowing the elements of  $\mathcal{B}$  to have different sizes. Let  $n, \mu, l$ , and  $\lambda$  be positive integers and  $K_\mu$  a finite set of positive integers such that  $n \geq \mu \geq l$  and such that  $\mu$  is the largest element in  $K_\mu$ . An  $l$ -( $n, K_\mu, \lambda$ ) *generalized covering*  $(V, \mathcal{B})$  is a pair  $(V, \mathcal{B})$  of a finite set  $V$  of cardinality  $n$  and a collection  $\mathcal{B}$  of subsets of  $V$  such that every  $l$ -subset of  $V$  appears in at least  $\lambda$  elements of  $\mathcal{B}$  and such that for any element  $B \in \mathcal{B}$ , the cardinality  $|B|$  is in  $K_\mu$ . When  $K_\mu$  is the singleton  $\{\mu\}$ , an  $l$ -( $n, K_\mu, \lambda$ ) generalized covering reduces to an  $l$ -( $n, \mu, \lambda$ ) covering. A generalized covering may also be seen as a straightforward generalization of a  $t$ -wise *balanced design*, where each  $l$ -subset of  $V$  occurs exactly  $\lambda$  times [25].

As in the standard covering number  $C_\lambda(n, \mu, l)$ , we define the *generalized covering number*  $C_\lambda(n, K_\mu, l)$  to be the cardinality of a smallest  $\mathcal{B}$  such that there exists an  $l$ -( $n, K_\mu, \lambda$ ) generalized covering  $(V, \mathcal{B})$ . We exploit the following lower bound on  $C_\lambda(n, K_\mu, l)$ .

**Proposition 3.1:** For any positive integers  $n, \mu, l, \lambda$  such that  $n \geq \mu \geq l$  and any set  $K_\mu$  of positive integers with  $\mu$  being its largest element,

$$C_\lambda(n, K_\mu, l) \geq \left\lceil \frac{n}{\mu} \left\lceil \frac{n-1}{\mu-1} \cdots \left\lceil \frac{\lambda(n-l+1)}{\mu-l+1} \right\rceil \right\rceil \right\rceil.$$

*Proof:* Let  $(V, \mathcal{B})$  be an  $l$ -( $n, K_\mu, \lambda$ ) generalized covering. For any element  $B \in \mathcal{B}$ , let  $B' = B \cup C_B$ , where  $C_B \subset V \setminus B$  is an arbitrary subset of cardinality  $\mu - |B|$ . Define  $\mathcal{B}' = \{B' \mid B \in \mathcal{B}\}$ . Then, the pair  $(V, \mathcal{B}')$  is an  $l$ -( $n, \mu, \lambda$ ) covering with  $|\mathcal{B}| = |\mathcal{B}'|$ , which implies that  $C_\lambda(n, K_\mu, l) \geq C_\lambda(n, \mu, l)$ . Applying the Schönheim bound for coverings given in [17] proves the assertion. ■

By noticing a simple relation between  $l$ -separating parity-check matrices and generalized coverings, we obtain the following lower bound on separating redundancy.

**Theorem 3.2:** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code with dual distance  $d^\perp$ . For any  $l \leq \min\{d, n-k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \geq \left\lceil \frac{n}{n-d^\perp} \left\lceil \frac{n-1}{n-d^\perp-1} \cdots \left\lceil \frac{\lambda(n-l+1)}{n-d^\perp-l+1} \right\rceil \right\rceil \right\rceil,$$

where  $\lambda = n - k - l$ .

*Proof:* Let  $H$  be an  $m \times n$   $l$ -separating parity-check matrix for  $\mathcal{C}$  and  $\mathbf{r}_i$  its  $i$ th row for  $0 \leq i \leq m-1$ . Regard each row  $\mathbf{r}_i = (r_0, \dots, r_{n-1})$ ,  $r_i \in \mathbb{F}_q$ , as an  $n$ -dimensional vector in  $\mathbb{F}_q^n$ . Let  $\text{supp}(\mathbf{r}_i) = \{j \mid r_j \neq 0, 0 \leq j \leq n-1\}$  be the complement of the support of  $\mathbf{r}_i$  and define  $\mathcal{B} = \{\text{supp}(\mathbf{r}_i) \mid 0 \leq i \leq m-1\}$ . Because  $H$  is  $l$ -separating, for any  $S \subset \{0, 1, \dots, n-1\}$  with  $|S| = l$ , there exists an  $m' \times n$  submatrix  $M_S$  of  $H$  such that  $\text{rank}(M_S) = n-k-l$  over  $\mathbb{F}_q$  and such that the  $m' \times |S|$  submatrix that consists of the columns indexed by  $S$  in  $M_S$  is the zero matrix. Thus, any  $S$  appears at least  $n-k-l$  times as a subset of an element of  $\mathcal{B}$ , which implies that the pair  $(\{0, 1, \dots, n-1\}, \mathcal{B})$  is an  $l$ -( $n, K_\mu, n-k-l$ )

generalized covering for some  $\mu \leq n - d^\perp$ . Hence, letting

$$L_\lambda(x, y, z) = \left\lceil \frac{x}{y} \left\lceil \frac{x-1}{y-1} \cdots \left\lceil \frac{\lambda(x-z+1)}{y-z+1} \right\rceil \right\rceil \right\rceil$$

for positive integers  $x, y, z$ , and  $\lambda$  with  $x \geq y \geq z$ , by Proposition 3.1, we have

$$\begin{aligned} s_l(\mathcal{C}) &\geq C_{n-k-l}(n, K_\mu, l) \\ &\geq L_l(n, \mu, n-k-l) \\ &\geq L_l(n, n-d^\perp, n-k-l), \end{aligned}$$

as desired. ■

It is notable that if we omit the ceiling functions in Theorem 3.2 to bound the right-hand side of the inequality from below, our lower bound reduces to the volume bound in Theorem 2.3. Hence, our bound is quite often sharper than Theorem 2.3 and always at least as sharp.

### B. Probabilistic upper bounds

We now turn our attention to upper bounds on the separating redundancy of a linear code. In what follows, for a pair  $x, y$  of nonnegative integers  $x \geq y$ ,

$$\begin{bmatrix} x \\ y \end{bmatrix}_q = \prod_{i=0}^{y-1} \frac{1 - q^{x-i}}{1 - q^{i+1}}$$

is defined to be the Gaussian binomial coefficient, which counts the number of  $y$ -dimensional subspaces in an  $x$ -dimensional subspace over  $\mathbb{F}_q$ .

To present our idea in a simple manner, we first prove the following basic upper bound through a probabilistic argument.

**Theorem 3.3:** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n-k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \leq \min_{t \in \mathbb{N}} \left\{ t + \left\lfloor \binom{n}{l} \sum_{r=0}^{n-k-l} (n-k-l-r) P_{t,r} \right\rfloor \right\},$$

where

$$P_{t,r} = \sum_{i=r}^t \binom{t}{i} (1 - q^{-l})^{t-i} \frac{\begin{bmatrix} n-k-l \\ r \end{bmatrix}_q \prod_{j=0}^{r-1} (q^i - q^j)}{q^{i(n-k)}}.$$

As we will later illustrate with numerical examples, the above basic bound can already be sharper than Theorem 2.4, which is the strongest among the known ones in many cases. Although it is not easy to directly compare Theorems 3.3 and 2.4 in general, to explain how our probabilistic approach is related to Theorem 2.4, we also prove an even sharper but slightly more complicated variant of Theorem 3.3. In addition to these, one more variant of Theorem 3.3 is given to present a slightly better upper bound for the case when the alphabet size  $q$  is large.

Now, to prove Theorem 3.3 and its two variants, we define a special kind of combinatorial matrix. An *orthogonal array*  $\text{OA}(m, n, g, l)$  is an  $m \times n$  matrix over a finite set  $\Gamma$  of cardinality  $g$  such that in any  $m \times l$  submatrix every  $l$ -dimensional vector in  $\Gamma^l$  appears exactly  $\frac{m}{g^l}$  times as a row. It is a well-known fact that an OA can be constructed by using

the codewords of a linear code as rows, which may be seen as a corollary of Delsarte's equivalence theorem [29, Theorem 4.5].

*Proposition 3.4:* Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code over  $\mathbb{F}_q$ . A  $q^{n-k} \times n$  matrix formed by all codewords of  $\mathcal{C}^\perp$  as rows is an OA( $q^{n-k}, n, q, d-1$ ).

We employ the following well-known fact.

*Lemma 3.5:* Take  $t, u, v \in \mathbb{N}$  with  $u \geq v$ . Let  $M$  be a  $t \times u$  matrix whose rows are drawn independently and uniformly at random from a  $v$ -dimensional subspace in a  $u$ -dimensional subspace over  $\mathbb{F}_q$ . For  $0 \leq r \leq v$ , the probability that  $M$  is of rank  $r$  over  $\mathbb{F}_q$  is

$$\frac{\begin{bmatrix} v \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^t - q^i)}{q^{vt}}.$$

Note that the above lemma is usually stated as an enumeration formula for  $t \times v$  matrices of rank  $r$  over a finite field. In fact, because a pair of subspaces of the same dimension over the same field are isomorphic, the  $u$ -dimensional ambient space in Lemma 3.5 is vacuous in the sense that we may as well consider  $t \times v$  matrices of a given rank with rows chosen from  $\mathbb{F}_q^v$ . However, because we apply a probabilistic argument to a linear code and its punctured code, we consider a subspace in a larger subspace and state the formula in probabilistic language. For various known proofs of Lemma 3.5, see, for example, [30], [31].

To present our proof in a concise manner, we slightly generalize the concept of an  $S$ -separating matrix by allowing the matrix under consideration to be a parity-check matrix for a supercode. Let  $A$  be a parity-check matrix for a supercode of a given  $[n, k, d]_q$  linear code  $\mathcal{C}$ . Define  $A(S)$  to be the submatrix of  $A$  obtained by discarding all rows with a nonzero element in at least one coordinate in  $S$  and deleting all columns corresponding to the coordinates in  $S$ . If  $S$  is the empty set, we define  $A(S) = A$ . The parity-check matrix  $A$  for the supercode is  $S$ -separating with respect to  $\mathcal{C}$  if  $A(S)$  is a parity-check matrix for  $\mathcal{C}_S$ . It is easy to see that for  $|S| \leq d-1$ , the matrix  $A$  is  $S$ -separating with respect to  $\mathcal{C}$  if and only if  $\text{rank}(A(S)) = n - k - |S|$ .

The following simple lemma plays a key role in our probabilistic argument.

*Lemma 3.6:* Let  $A$  be a parity-check matrix for a supercode of an  $[n, k, d]_q$  linear code  $\mathcal{C}$  and take a positive integer  $l \leq \min\{d, n-k\} - 1$ . If  $A$  is  $S$ -separating with respect to  $\mathcal{C}$  for any  $S \subset \{0, 1, \dots, n-1\}$  of cardinality  $l$ , then  $A$  is an  $l$ -separating parity-check matrix for  $\mathcal{C}$ .

*Proof:* Let  $S' \subset \{0, 1, \dots, n-1\}$  be a set of  $l-1$  coordinates. Because  $\text{rank}(A(S')) = n - k - l + 1 > 0$  for any  $S' \subset \{0, 1, \dots, n-1\}$  of cardinality  $l-1$ , the matrix  $A(S')$  contains at least one nonzero element. Without loss of generality, we assume that the  $i$ th column of  $A$  contains a nonzero element in a row of  $A(S')$ , where  $i \notin S'$ . Because  $A$  is  $(S' \cup \{i\})$ -separating with respect to  $\mathcal{C}$  by assumption, we have  $\text{rank}(A(S' \cup \{i\})) = n - k - l$ . Thus, since  $i \notin S'$  and the  $i$ th column of  $A$  contains a nonzero element in a row of  $A(S')$ , we have  $\text{rank}(A(S')) \geq n - k - l + 1$ . However, because  $A$  is a parity-check matrix for a supercode of  $\mathcal{C}_S$ , we have  $\text{rank}(A(S')) \leq n - k - l + 1$ , which implies that

$\text{rank}(A(S')) = n - k - l + 1$ . Hence,  $A$  is  $S'$ -separating with respect to  $\mathcal{C}$ . By induction, for any  $S'' \subset \{0, 1, \dots, n-1\}$  of cardinality at least 1, the matrix  $A$  is  $S''$ -separating with respect to  $\mathcal{C}$ . Consider a singleton  $\{j\} \subset \{0, 1, \dots, n-1\}$ . Because  $A$  is  $\{j\}$ -separating with respect to  $\mathcal{C}$ , we have  $\text{rank}(A(\{j\})) = n - k - 1$ . If the  $j$ th column of  $A$  contains a nonzero element, then  $A$  is of rank  $n - k$  and thus a parity-check matrix for  $\mathcal{C}$ . Therefore, if there exists a column of  $A$  that contains a nonzero element, then we are done. If all columns are zero column vectors, then  $A$  is the zero matrix, which contradicts the assumption that  $A$  is  $S$ -separating with respect to  $\mathcal{C}$  for any  $S \subset \{0, 1, \dots, n-1\}$  of cardinality  $l$ . ■

We are now ready to present the probabilistic proof of our basic upper bound. In what follows, the expected value of a given random variable  $X$  is denoted by  $\mathbb{E}(X)$ .

*Proof of Theorem 3.3:* Construct a  $t \times n$  matrix  $A$  by taking independently and uniformly at random  $t$  codewords from  $\mathcal{C}^\perp$  as rows. Let  $\mathcal{S} = \{S \subset \{0, 1, \dots, n-1\} \mid |S| = l\}$  be the set of  $\binom{n}{l}$  subsets  $S \subset \{0, 1, \dots, n-1\}$  of cardinality  $l$ . By Proposition 3.4 and Lemma 3.5, for any  $S \in \mathcal{S}$ , the probability  $p_{A,S,r}$  that  $A(S)$  is of rank  $r$  is

$$\begin{aligned} p_{A,S,r} &= \sum_{i=r}^t \binom{t}{i} q^{-li} (1 - q^{-l})^{t-i} \frac{\begin{bmatrix} n-k-l \\ r \end{bmatrix}_q \prod_{j=0}^{r-1} (q^i - q^j)}{q^{i(n-k-l)}} \\ &= P_{t,r}. \end{aligned}$$

We adjoin more rows if  $\text{rank}(A(S))$  is less than  $n - k - l$ . Let  $X$  be the random variable counting the smallest number of additional rows required to attach to  $A$  to turn it into an  $S$ -separating matrix with respect to  $\mathcal{C}$  for any  $S \in \mathcal{S}$ . Trivially,  $X \leq \mathbb{E}(X)$ . Thus, by Lemma 3.6, appending appropriately chosen rows to  $A$  gives an  $l$ -separating parity-check matrix for  $\mathcal{C}$  with at most  $t + \lfloor \mathbb{E}(X) \rfloor$  rows. Therefore, we have

$$s_l(\mathcal{C}) \leq \min_{t \in \mathbb{N}} \{t + \lfloor \mathbb{E}(X) \rfloor\}. \quad (1)$$

To bound the expected value  $\mathbb{E}(X)$  on the right-hand side from above, notice that

$$\begin{aligned} \mathbb{E}(X) &\leq \mathbb{E} \left( \sum_{S \in \mathcal{S}} (n - k - l - \text{rank}(A(S))) \right) \\ &= \binom{n}{l} \sum_{r=0}^{n-k-l} (n - k - l - r) P_{t,r}. \end{aligned}$$

Plugging in the above upper bound into Inequality (1) completes the proof. ■

While we focused on a simple presentation of our idea, the probabilistic upper bound in Theorem 3.3 may be improved by more careful analyses. As stated earlier, we present two useful variants that do not require too involved an argument.

Recall that Theorem 2.4 involves the function  $f_q(a, b)$  for nonnegative integers  $a, b$  and a prime power  $q$ . For  $b \leq a$ , it was proved in [13] that this counts the number of  $a \times b$  matrices over  $\mathbb{F}_q$  of rank  $b$  that do not contain all-zero rows. We incorporate this knowledge to replace in our probabilistic argument the elementary fact, which is Lemma 3.5, with the following lemma that allows for using a slightly more favorable probability space.

**Lemma 3.7:** Take  $t, u, v \in \mathbb{N}$  with  $u \geq v$ . Let  $\mathcal{E}$  be a  $v$ -dimensional subspace in a  $u$ -dimensional vector space over  $\mathbb{F}_q$  and  $M$  a  $t \times u$  matrix whose rows are drawn independently and uniformly at random from  $\mathcal{E} \setminus \{0\}$ . For  $0 \leq r \leq v$ , the probability that  $M$  is of rank  $r$  over  $\mathbb{F}_q$  is

$$\frac{\begin{bmatrix} v \\ r \end{bmatrix}_q f_q(t, r)}{(q^v - 1)^t}.$$

*Proof:* For any  $r$ -dimensional subspace  $\mathcal{F}$  of  $\mathcal{E}$ , take an  $r \times u$  matrix  $B_{\mathcal{F}}$  whose rows form a basis of  $\mathcal{F}$ . Every  $t \times u$  matrix  $N$  of rank  $r$  over  $\mathbb{F}_q$  without all-zero rows can be written uniquely as a product  $N = RB_{\mathcal{F}}$  of two matrices for some  $\mathcal{F}$ , where  $R$  is a  $t \times r$  matrix of rank  $r$  over  $\mathbb{F}_q$  without all-zero rows. Hence,  $N$  and the pair  $(R, B_{\mathcal{F}})$  has a one-to-one correspondence. Because  $M$  is taken uniformly at random from all possible  $(q^v - 1)^t$  matrices, the claim follows by dividing the number of pairs  $(R, B_{\mathcal{F}})$  by  $(q^v - 1)^t$ . ■

Because the zero codeword  $0$  in the dual code  $\mathcal{C}^\perp$  does not contribute to anything when present in a parity-check matrix for  $\mathcal{C}$  other than inflating the number of rows, use of the above lemma leads to a better upper bound.

**Theorem 3.8:** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n - k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \leq \min_{t \in \mathbb{N}} \left\{ t + \left\lfloor \binom{n-k-l}{l} \sum_{r=0}^{n-k-l} (n-k-l-r) Q_{t,r} \right\rfloor \right\},$$

where

$$Q_{t,r} = \sum_{i=r}^t \binom{t}{i} c^i (1-c)^{t-i} \frac{\begin{bmatrix} n-k-l \\ r \end{bmatrix}_q f_q(i, r)}{(q^{n-k-l} - 1)^i}$$

with  $c = \frac{q^{n-k-l}-1}{q^{n-k}-1}$ .

*Proof:* Construct a  $t \times n$  matrix  $A$  by taking independently and uniformly at random  $t$  codewords from  $\mathcal{C}^\perp \setminus \{0\}$  as rows. Argue in the same manner as in the proof of Theorem 3.3 by using Lemma 3.7 in place of Lemma 3.5. ■

To see how Theorem 3.8 is related to Theorem 2.4, let  $e(t, \mathcal{C})$  and  $e(t, \mathcal{C}, S)$  be the number of  $t \times n$  parity-check matrices with no all-zero rows for a given  $[n, k, d]_q$  linear code  $\mathcal{C}$  and that of  $S$ -separating ones with no all-zero rows for a given subset  $S \subset \{0, 1, \dots, n-1\}$  respectively. The proof of Theorem 2.4 calculates  $e(t, \mathcal{C})$  and  $e(t, \mathcal{C}, S)$ , which also shows that  $e(t, \mathcal{C}, S) = e(t, \mathcal{C}, S')$  for any pair  $S, S'$  with  $|S| = |S'|$ . Hence, for any subset  $S \subset \{0, 1, \dots, n-1\}$  of cardinality  $l$ , we may safely write  $e(t, \mathcal{C}, l)$  to mean the number of  $S$ -separating parity-check matrices for  $\mathcal{C}$  with  $t$  rows. With this notation, the pigeonhole principle ensures that there exists at least one  $t \times n$   $l$ -separating parity-check matrix for  $\mathcal{C}$  if

$$e(t, \mathcal{C}) > \binom{n}{l} (e(t, \mathcal{C}) - e(t, \mathcal{C}, l)). \quad (2)$$

Theorem 2.4 is a claim that the  $l$ -separating redundancy of  $\mathcal{C}$  must be smaller than or equal to the smallest possible  $t$  that satisfies the above inequality.

It is well-known that a counting argument of this kind can be translated into a probabilistic one either through the union

bound or through linearity of expectation. For our purpose, it is more convenient to choose the latter.

Let  $\mathcal{S} = \{S \subset \{0, 1, \dots, n-1\} \mid |S| = l\}$  be the set of  $\binom{n}{l}$  subsets  $S \subset \{0, 1, \dots, n-1\}$  of cardinality  $l$ . Take a  $t \times n$  parity-check matrix  $H$  uniformly at random from the set of  $e(t, \mathcal{C})$  parity-check matrices with no all-zero rows for  $\mathcal{C}$ . Define  $X_S$  to be the random variable that equals 0 if  $H$  is  $S$ -separating and 1 otherwise. Let  $X = \sum_{S \in \mathcal{S}} X_S$ . If  $\mathbb{E}(X) < 1$ , then there exists a  $t \times n$   $l$ -separating parity-check matrix, which implies that if

$$\begin{aligned} 1 &> \mathbb{E}(X) \\ &= \sum_{S \in \mathcal{S}} \mathbb{E}(X_S) \\ &= \binom{n}{l} \left( 1 - \frac{e(t, \mathcal{C}, l)}{e(t, \mathcal{C})} \right), \end{aligned}$$

then there exists an  $l$ -separating parity-check matrix for  $\mathcal{C}$  with  $t$  rows. Clearly, the above inequality is equivalent to Inequality (2), proving the same bound by a probabilistic argument.

In the proofs of Theorems 3.3 and 3.8, we randomly sample a parity-check matrix  $A$  for a supercode of  $\mathcal{C}$ . Note that if  $A$  has a large number of rows, it is very likely  $l$ -separating. Hence, if we took sufficiently large  $A$ , then the argument would be nearly identical to the probabilistic version of the proof of Theorem 2.4 except that with our approach there would be a very tiny probability that the chosen  $A$  lacked enough linearly independent rows to be a parity-check matrix for  $\mathcal{C}$ . The trick we used for our upper bounds is that we deliberately pick rather small  $A$  and, if it is not  $S$ -separating for some  $S \in \mathcal{S}$ , fix the blemishes by appending a few more rows. Once we finish making our small matrix  $S$ -separating for any  $S \in \mathcal{S}$ , Lemma 3.6 assures that the modified matrix is automatically a parity-check matrix for  $\mathcal{C}$  rather than for its supercode.

The crucial point is that, unlike the probabilistic proof of Theorem 2.4, the proofs of Theorems 3.3 and 3.8 do not require a probability space in which a randomly chosen matrix is typically  $l$ -separating. Hence, our bounds can be sharper as long as we can make a good estimate of the required number of additional rows.

One weakness of Theorems 3.3 and 3.8 is that these probabilistic bounds tend to give larger estimates as the alphabet size  $q$  increases. This contrasts with the fact that the lower bound in Theorem 3.2 is independent of  $q$ . In the remainder of this subsection, we show that a simple trick can give an alternative bound that mitigates this weakness to an extent.

**Theorem 3.9:** Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n - k\} - 1$ , the  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \leq n - k + \min_{t \in \mathbb{N}} \left\{ t + \left\lfloor b_l \sum_{r=0}^{n-k-l} (n-k-l-r) Q_{t,r} \right\rfloor \right\},$$

where  $b_l = \binom{n}{l} - \binom{n-k}{l}$  and

$$Q_{t,r} = \sum_{i=r}^t \binom{t}{i} c^i (1-c)^{t-i} \frac{\begin{bmatrix} n-k-l \\ r \end{bmatrix}_q f_q(i, r)}{(q^{n-k-l} - 1)^i}$$



with  $c = \frac{q^{n-k-l}-1}{q^{n-k}-1}$ .

*Proof:* Take an  $(n-k) \times n$  parity-check matrix  $H$  for  $\mathcal{C}$  in standard form, so that  $H$  contains the  $(n-k) \times (n-k)$  identity matrix  $I$  as its submatrix. Let  $T$  be the set of coordinates that index the columns of  $I$  in  $H$ . Because  $I$  contains exactly one nonzero element in each column and in each row, we have  $\text{rank}(H(S)) = n - k - l$  for any  $l$ -subset  $S \subset T$ . Hence,  $H$  is  $S$ -separating for any  $S \subset T$  of cardinality  $l$ . Carrying out the same argument as in the proof of Theorem 3.8 over the remaining  $\binom{n}{l} - \binom{n-k}{l}$   $l$ -subsets of coordinates proves the assertion. ■

Theorem 3.9 reduces the coefficient of the sum from  $\binom{n}{l}$  to  $\binom{n}{l} - \binom{n-k}{l}$  in exchange for the newly introduced constant term  $n - k$ . While the additional  $n - k$  rows is a nontrivial penalty, as we will illustrate with a nonbinary linear code, the benefit of the smaller coefficient outweighs the disadvantage in the constant term when the sum of  $rQ_{r,t}$  over all  $r$  is large.

### C. Refined design theoretic upper bound

Probabilistic combinatorics provides powerful tools for proving the existence of a desired mathematical object. However, verifying the existence alone does not necessarily supply an efficient algorithm for construction. While our probabilistic upper bounds are general and quite strong compared to the other known general upper bounds, their proofs do not give any insight into how we may be able to efficiently construct  $l$ -separating parity-check matrices that achieve the bounds.

To address this disadvantage, we refine the design theoretic upper bound given in Theorem 2.6. As long as a suitable covering can be constructed efficiently, our design theoretic bound is constructive and, in some cases, sharper than the probabilistic ones.

*Theorem 3.10:* Let  $\mathcal{C}$  be an  $[n, k, d]_q$  linear code. For any  $l \leq \min\{d, n - k\} - 1$ , define  $L = \{i \in \mathbb{N} \mid l \leq i \leq \min\{d, n - k\} - 1\}$ . The  $l$ -separating redundancy  $s_l(\mathcal{C})$  satisfies

$$s_l(\mathcal{C}) \leq \min_{\mu \in L} \left\{ (n - k)C_1(n, \mu, l), (n - k - l) \binom{n}{l} \right\}.$$

*Proof:* Take a positive integer  $\mu \in L$ . Let  $(\{0, 1, \dots, n - 1\}, \mathcal{B})$  be an  $l$ -( $n, \mu, 1$ ) covering and  $H_{\text{all}}$  a  $q^{n-k} \times n$  parity-check matrix for  $\mathcal{C}$  whose rows are the  $q^{n-k}$  codewords of  $\mathcal{C}^\perp$ . By Proposition 3.4, for any  $\mu$ -subset  $S \subset \{0, 1, \dots, n - 1\}$ , the parity-check matrix  $H_{\text{all}}$  contains a  $\mu \times n$  submatrix  $I_S$  whose columns indexed by the elements of  $S$  form the  $\mu \times \mu$  identity matrix. Because  $\text{rank}(H_{\text{all}}) = n - k$ , it also contains an  $(n - k - \mu) \times n$  submatrix  $M_S$  of rank  $n - k - \mu$  over  $\mathbb{F}_q$  whose columns indexed by the elements of  $S$  form the zero matrix. It is straightforward to see that the  $(n - k) \times n$  matrix  $H_S$  obtained by stacking  $I_S$  on top of  $M_S$  is an  $S$ -separating parity-check matrix for  $\mathcal{C}$ . Taking  $H_B$  for all  $B \in \mathcal{B}$  and stacking them on top of each other gives an  $l$ -separating parity-check matrix with  $(n - k)|\mathcal{B}|$  rows. Therefore,  $s_l(\mathcal{C}) \leq (n - k)C_1(n, \mu, l)$ . To obtain an  $l$ -separating parity-check matrix with  $(n - k - l)\binom{n}{l}$  rows, stack  $M_S$  for all  $S \subset \{0, 1, \dots, n - 1\}$  of cardinality  $l$ . ■

To see how Theorem 3.10 improves Theorem 2.6, consider the following proof of the latter. First, as in the proof of Theorem 3.10, take an  $l$ -( $n, \mu, 1$ ) covering  $(\{0, 1, \dots, n - 1\}, \mathcal{B})$

and stack  $M_S$  for  $S \in \mathcal{B}$  on top of each other. This ensures that the resulting matrix  $H$  satisfies  $\text{rank}(H(S')) \geq n - k - \mu$  for any  $l$ -subsets  $S' \subset \{0, 1, \dots, n - 1\}$ . To turn  $H$  into an  $l$ -separating one, we may simply adjoin a  $(\mu - l) \times n$  matrix  $I_{S'}$  for all possible  $\binom{n}{l}$  patterns of  $S'$ , which establishes the same upper bound as in Theorem 2.6.

Note that while we can use an arbitrary covering in the first step of the above alternative proof, the second step uses the poorest, trivial covering that consists of the  $\binom{n}{l}$   $l$ -subsets themselves. The key idea in the proof of Theorem 3.10 is that we can also exploit the same nontrivial covering in the second step so that we do not need to fix each of the  $\binom{n}{l}$  blemishes one-by-one. Indeed, use of a covering with  $|\mathcal{B}| < \frac{\mu-l}{\mu} \binom{n}{l}$  always results in a smaller  $l$ -separating parity-check matrix.

The benefit of the refinement we described above disappears if  $l = \mu = \min\{d, n - k\} - 1$ . In this degenerate case, the first step alone gives an  $l$ -separating parity-check matrix, so that Theorems 3.10 and 2.6 both reduce to the same bound  $s_l(\mathcal{C}) \leq (n - k - l)\binom{n}{l}$ .

It is notable that we can also exploit any  $l$ -( $n, K_\mu, 1$ ) generalized covering  $(\{0, 1, \dots, n - 1\}, \mathcal{B})$ , even if  $\mu \geq d$  and regardless of whether  $K_\mu$  is a singleton, as long as for any  $B \in \mathcal{B}$ , the columns indexed by the elements of  $B$  in a parity-check matrix for  $\mathcal{C}$  are linearly independent. Indeed, linear independence among the columns indexed by the elements of  $B$  ensures that  $H_{\text{all}}$  contains the key components  $I_B$ ,  $M_B$ , where  $I_B$  is a  $|B| \times n$  submatrix whose columns indexed by the elements of  $B$  form the identity matrix and  $M_B$  is an  $(n - k - |B|) \times n$  submatrix of rank  $n - k - |B|$  over  $\mathbb{F}_q$  whose columns indexed by the elements of  $B$  form the zero matrix. Hence, as in the proof of Theorem 3.10, we can obtain an  $l$ -separating parity-check matrix by stacking  $I_B$  and  $M_B$  for all  $B \in \mathcal{B}$ .

To illustrate the generalized approach, we apply it to a class of linear codes from finite geometry. For fundamental notions and basic facts in finite geometry, we refer the reader to [32].

The *affine geometry*  $\text{AG}(m, q)$  of dimension  $m$  over  $\mathbb{F}_q$  is a finite geometry whose *points* are the vectors in  $\mathbb{F}_q^m$  and  *$i$ -flats* are the  $i$ -dimensional vector spaces of  $\mathbb{F}_q^m$  and their cosets. We use  $\text{AG}(2, q)$  with  $q$  even and consider its points and 1-flats, that is, the *affine plane* with  $q = 2^h$  and its  $4^h$  points and  $2^h(4^h - 1)$  *lines*. We assume that the points and lines are both arbitrarily ordered.

The *line-by-point incidence matrix*  $H = (h_{i,j})$  is the  $2^h(4^h - 1) \times 4^h$  matrix over  $\mathbb{F}_2$  whose rows and columns are indexed by the lines and points respectively such that the entry  $h_{i,j}$  of the  $i$ th row of the  $j$ th column is 1 if the  $i$ th line passes through the  $j$ th point and 0 otherwise. It is known that the binary linear code defined by  $H$  as a parity-check matrix is of parameters  $[4^h, 4^h - 3^h, 2^h + 2]_2$  (see [33] for the earliest proof and also [34] for the same fact in our terminology). Using a generalized covering tailored to this linear code, we construct a smaller 5-separating parity-check matrix than is achievable by Theorem 3.10.

*Theorem 3.11:* Let  $H$  be the line-by-point incidence matrix of affine geometry  $\text{AG}(2, 2^h)$  with  $h \geq 3$ . Define  $\mathcal{C}$  to be the  $[4^h, 4^h - 3^h, 2^h + 2]_2$  linear code obtained by using  $H$  as a parity-check matrix. Then, the 5-separating redundancy of  $\mathcal{C}$

satisfies

$$s_5(\mathcal{C}) \leq 3^h(2^{5h} + 3 \cdot 2^{4h-1} + 9 \cdot 2^{3h-1} - 3 \cdot 2^h).$$

*Proof:* Let  $\mathcal{L}$  be the set of lines in  $\text{AG}(2, 2^h)$ . For any pair  $l_0, l_1 \in \mathcal{L}$  of parallel lines, take an arbitrary set  $U_{l_0, l_1}^6$  of six parallel lines each of which intersects both  $l_0$  and  $l_1$ . Define  $\mathcal{L}_0$  to be the set of irreducible conics in  $\text{AG}(2, 2^h)$ . Let  $\mathcal{L}_1 = \{l_0 \cup l_1 \mid l_0, l_1 \in \mathcal{L}, l_0 \text{ and } l_1 \text{ are nonparallel}\}$  and  $\mathcal{L}_2 = \{(l_0 \cup l_1) \setminus l_2 \mid l_0, l_1 \in \mathcal{L} \text{ are parallel}, l_2 \in U_{l_0, l_1}^6\}$ . Define  $\mathcal{B} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2$ . It is routine to show that  $|\mathcal{L}_0| = 2^{5h} + 2^{4h} + 2^{3h}$ ,  $|\mathcal{L}_1| = \binom{2^h+1}{2} 2^{2h}$ , and  $|\mathcal{L}_2| = 6(2^h+1)\binom{2^h}{2}$ . Hence, because there is no overlap between  $\mathcal{L}_0$ ,  $\mathcal{L}_1$ , and  $\mathcal{L}_2$ , we have  $|\mathcal{B}| = 2^{5h} + 3 \cdot 2^{4h-1} + 9 \cdot 2^{3h-1} - 3 \cdot 2^h$ .

We first show that every quintuple of points appears in an element of  $\mathcal{B}$ . Let  $S$  be a set of five points in  $\text{AG}(2, 2^h)$ . If the five points in  $S$  all lie on a single line  $l_0$ , there exists another line  $l_1 \neq l_0$  that passes through a point  $p \notin S$  on  $l_0$ . Hence,  $S$  appears in  $l_0 \cup l_1 \in \mathcal{L}_1$ . By the same token, if exactly four points in  $S$  are collinear, considering a line  $l_1$  that passes through the remaining point in  $S$  and intersects the line  $l_0$  that carries the four points, we have  $S \subset l_0 \cup l_1 \in \mathcal{L}_1$ . If no three points in  $S$  are collinear, because  $|S| = 5$ , the set  $S$  is contained in exactly one irreducible conic in  $\mathcal{L}_0$ . The remaining case is when no four points in  $S$  are collinear while  $S$  contains three points on the same line. We consider two subcases.

*Case 1.* There is exactly one subset  $\{p_0, p_1, p_2\} \subset S$  of three collinear points. Let  $l_0$  be the line that carries  $\{p_0, p_1, p_2\}$ . If the line  $l_1$  that carries the remaining two points in  $S$  intersects  $l_0$ , then  $S$  appears in  $l_0 \cup l_1 \in \mathcal{L}_1$ . If  $l_1$  is parallel to  $l_0$ , then  $S$  appears in an element  $(l_0 \cup l_1) \setminus l_2$  of  $\mathcal{L}_2$ , where  $l_2 \in U_{l_0, l_1}^6$  is a line that contains no point in  $S$ .

*Case 2.* There is more than one subset of three collinear points in  $S$ . Because  $|S| = 5$ , we have exactly two subsets  $S_0, S_1$  of three collinear points in  $S$ . Let  $l_0$  and  $l_1$  be the lines that carry  $S_0$  and  $S_1$  respectively. Because  $l_0$  and  $l_1$  are nonparallel,  $S$  appears in  $l_0 \cup l_1 \in \mathcal{L}_1$ .

It now suffices to prove that for any  $B \in \mathcal{B}$ , the columns of  $H$  indexed by the elements of  $B$  are linearly independent. We show that every  $B \in \mathcal{B}$  has a tangent at any point in  $B$ . If  $B \in \mathcal{L}_0$ , it is an irreducible conic and hence has a tangent at any point in  $B$  as desired. If  $B = l_0 \cup l_1 \in \mathcal{L}_1$  for nonparallel lines  $l_0, l_1$ , both  $l_0$  and  $l_1$  are tangents at their intersection  $p$ , while any point  $p' \neq p$  on  $l_0$  or  $l_1$  lies on a line parallel to  $l_1$  or  $l_0$  respectively, which is a tangent at  $p'$  on  $B$ . Finally, if  $B = (l_0 \cup l_1) \setminus l_2$  for a pair  $l_0, l_1$  of parallel lines and  $l_2 \in U_{l_0, l_1}^6$ , the line that passes through a point  $p$  on  $l_0 \setminus l_2$  and the intersection of  $l_1$  with  $l_2$  is a tangent at  $p$ . By symmetry, the line that passes through a point  $p'$  on  $l_1 \setminus l_2$  and the intersection of  $l_0$  with  $l_2$  is a tangent at  $p'$ . ■

Considering how small a covering can be, it is straightforward to see that for  $l \leq d-2$ , the right-hand side of the bound in Theorem 3.10 is lower bounded by

$$\min_{\mu \in \mathcal{L}} \left\{ (n-k)C_1(n, \mu, l), (n-k-l)\binom{n}{l} \right\} \geq (n-k)\frac{\binom{n}{l}}{\binom{d-1}{l}}.$$

For the linear code of length  $n = 4^h$  and minimum distance

$d = 2^h + 2$ , we have

$$\begin{aligned} \frac{\binom{n}{5}}{\binom{d-1}{5}} &= \frac{2^h(2^h+2)(4^h-2)(4^h-3)}{2^h-3} \\ &> 2^{5h} + 3 \cdot 2^{4h-1} + 9 \cdot 2^{3h-1} - 3 \cdot 2^h. \end{aligned}$$

This shows that Theorem 3.11 provides a smaller 5-separating parity-check matrix due to the use of a more efficient generalized covering than the standard ones considered in Theorem 3.10.

#### IV. NUMERICAL EXAMPLES

In this section, we illustrate how our bounds compare against the known ones in specific cases by numerically bounding the separating redundancies of some short exemplary linear codes. This complements the technical details on the refinements explained from the general viewpoint in the previous section.

Tables I, II, and III list numerical results on the lower and general upper bounds for the  $[24, 12, 8]_2$  binary Golay code,  $[41, 33, 5]_3$  ternary Bose-Chaudhuri-Hocquenghem (BCH) code, and  $[12, 6, 6]_4$  quaternary quadratic residue code respectively. As expected from the theoretical analyses in the previous section, our theorems consistently provide strong bounds and quite often improve the sharpest known results. In particular, our basic probabilistic upper bound by Theorem 3.3 is already consistently strong, while Theorems 3.8 and 3.9 improve this result even further in some cases. Our lower bound by Theorem 3.2 also provides a solid improvement in many cases while being always at least as tight as the strongest known one in the literature.

It is notable that, as can be seen in Table III, the design theoretic approach taken in Theorems 3.10 and 2.6 can occasionally result in upper bounds that surpass all other known ones, which is a fact that does not seem to have been emphasized in the literature. We also draw attention with this  $[12, 6, 6]_4$  linear code to the fact that while Theorem 3.10 greatly improves Theorem 2.6 in the vast majority of the cases, there are instances where it fails to be even as sharp. This rare situation occurs when the right-hand side of the bound in Theorem 3.10 is minimized by  $\mu$  such that  $C_1(n, \mu, l) > \frac{\mu-l}{\mu}\binom{n}{l}$ .

For completeness, it is worth mentioning that the specialized bound given in [13, Corollary 6] for the 1-separating redundancy  $s_1(\mathcal{C})$  of a linear code  $\mathcal{C}$  with the all-one codeword in the dual code  $\mathcal{C}^\perp$  gives  $s_1(\mathcal{C}) \leq 22$  for the Golay code and  $s_1(\mathcal{C}) \leq 20$  for the quadratic residue code. Except for these two cases, no known specialized bounds improve or match the best results in the tables by our general theorems.

#### V. CONCLUDING REMARKS

We have presented various new general bounds on the separating redundancy of a linear code. Progress has been made both on the lower and on the upper bounds through probabilistic combinatorics and design theory.

The lower bound we gave is always at least as sharp as the previously known one by Theorem 2.3 and quite often sharper. It is notable that there exist linear codes that achieve our



TABLE I  
BOUNDS ON THE  $l$ -SEPARATING REDUNDANCY OF THE  $[24, 12, 8]_2$  BINARY GOLAY CODE

Bound	Type	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$	$l = 7$
Theorem 3.2	lower	17	24	35	50	75	114	162
Theorem 2.3 [13]	lower	17	23	33	47	69	101	152
Theorem 3.3	upper	35	84	185	386	781	1539	2970
Theorem 3.8	upper	35	84	185	386	780	1539	2969
Theorem 3.9	upper	44	94	195	397	791	1550	2980
Theorem 3.10 <sup>a</sup>	upper	48	204	936	—	—	—	—
Theorem 2.4 [13] <sup>a b</sup>	upper	37	93	214	466	984	2034	—
Theorem 2.5 [13]	upper	78	298	793	1585	2509	3301	3796
Theorem 2.6 [13] <sup>a</sup>	upper	120	936	—	—	—	—	—

<sup>a</sup> The cases when these upper bounds become weaker than the trivial one  $s_l(C) < q^{n-k}$  for a linear code  $C$  of length  $n$  and dimension  $k$  over  $\mathbb{F}_q$  are marked by “—”.

<sup>b</sup> Errors in Table I in [13] are corrected.

TABLE II  
BOUNDS ON THE  $l$ -SEPARATING REDUNDANCY OF THE  $[41, 33, 5]_3$  TERNARY BCH CODE

Bound	Type	$l = 1$	$l = 2$	$l = 3$	$l = 4$
Theorem 3.2 <sup>a</sup>	lower	16	35	71	146
Theorem 2.3 [13] <sup>a</sup>	lower	16	33	66	133
Theorem 3.3	upper	37	137	445	1366
Theorem 3.8	upper	37	137	445	1366
Theorem 3.9	upper	44	144	452	1374
Theorem 3.10 <sup>b</sup>	upper	48	1152	—	—
Theorem 2.4 [13]	upper	40	160	558	1836
Theorem 2.5 [13]	upper	64	288	848	1744
Theorem 2.6 [13] <sup>b</sup>	upper	113	2190	—	—

<sup>a</sup> For the dual distance, the upper bound found in [35] is used.

<sup>b</sup> The cases when these upper bounds become weaker than the trivial one  $s_l(C) < q^{n-k}$  for a linear code  $C$  of length  $n$  and dimension  $k$  over  $\mathbb{F}_q$  are marked by “—”.

TABLE III  
BOUNDS ON THE  $l$ -SEPARATING REDUNDANCY OF THE  $[12, 6, 6]_4$  QUATERNARY QUADRATIC RESIDUE CODE

Bound	Type	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$
Theorem 3.2	lower	10	18	36	66	132
Theorem 2.3 [13]	lower	10	18	33	66	132
Theorem 3.3	upper	29	112	351	823	792
Theorem 3.8	upper	29	112	351	822	792
Theorem 3.9	upper	30	111	346	815	792
Theorem 3.10	upper	30	54	174	678	792
Theorem 2.4 [13] <sup>a</sup>	upper	34	166	688	2622	—
Theorem 2.5 [13]	upper	51	231	636	1122	1365
Theorem 2.6 [13]	upper	48	138	334	608	792

<sup>a</sup> When  $l = 5$ , Theorem 2.4 is weaker than the trivial bound  $s_l(C) < q^{n-k}$  for a linear code  $C$  of length  $n$  and dimension  $k$  over  $\mathbb{F}_q$  and is marked by “—”.

lower bound by Theorem 3.2 for some  $l$ . Interesting examples include all MDS codes with  $l = d - 2$ , where Theorems 3.2 and 2.3 coincide and are both achieved by the  $[n, k, n - k + 1]_q$  codes (see [13] for the  $(d - 2)$ -separating redundancy of an MDS code). Thus, any general lower bound which is at least as sharp must reduce to Theorem 3.2 in the achievable cases. For this reason, it appears very difficult to give a simple and better lower bound without imposing some condition on the applicable linear codes, the range of  $l$ , or both.

To also bound on separating redundancy from above, we refined two known approaches and proved general bounds that are applicable even when little structural information is available other than the basic code parameters. Through theoretical analyses and numerical examples, our theorems were shown to be much sharper than the previously known bounds in many cases.

It should be noted, however, that there is still a considerable gap between the best upper and lower bounds in general. Although we expect that a fundamentally different approach is required to substantially improve our bounds, minor improvements might be possible within the same framework through, for instance, the probabilistic method by random sampling without replacement and/or a more careful analysis on the required number of rows to fix all blemishes. It would be an interesting combinatorial problem on its own to bound separating redundancy as tightly as possible in the general case.

It should also be noted that, from a more purely coding theoretic point of view, it may be of equal or, perhaps, greater importance to investigate particular error-correcting codes and their practical implementation in the context of error-erasure separation. Indeed, research in this direction can be found in the literature for a class of geometric low-density parity-check codes [36]. With the progress on separating redundancy we have made in the general case, we believe that more specialized approaches tailored to specific linear codes and research on practical implementation also deserve greater attention in the future work.

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